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CONSERVATIVE QUANTITES OF FREE PARTICLE IN THE INTERIOR SCHWARZSCHILD SPACE-TIME

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Abstract—The interior Schwarzschildsolution of Einstein field equations in presence of matter is considered. The non zerochristoffel symbols of such a solutions are obtained, these symbols are used to obtain some conservative quantities in some spaces in the geometry of such a solutions. The Killing equations are solved in sub spaces of the interior Schwarzschild space time under some condition.

Index Terms— The Killing equations, conservative quantities, interior Schwarzschild solution,Schwarzschild.

1 Introduction

The necessary and sufficient condition for existence of symmetry of space time relative to a vector field ξ^α is [1];

$$\xi_{\mu\|\nu} + \xi_{\nu\|\mu} = 0 \qquad (1.1)$$

Which can written in the form,

$$\frac{\partial \xi_\mu}{\partial x^\nu} + \frac{\partial \xi_\nu}{\partial x^\mu} - 2\xi_\alpha \left\{ {}_{\mu}{}^{\alpha}{}_{\nu} \right\} = 0. \qquad (1.2)$$

The interior Schwarzschild space-time metric is defined by [2];

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$$ds^2 = \left[\frac{3}{2}\left(1 - \frac{r_0^2}{\hat{R}^2}\right)^{\frac{1}{2}} - \frac{1}{2}\left(1 - \frac{r^2}{\hat{R}^2}\right)^{\frac{1}{2}}\right]^2 c^2\, dt^2 - \left(1 - \frac{r^2}{\hat{R}^2}\right)^{-1} dr^2 \;\; - r^2 d\theta^2 - r^2 \sin^2\theta\, d\varphi^2, \qquad (1.3)$$

where, $\hat{R}^2 = \frac{3c^2}{8\pi G\rho}$, G is the gravitational constant, ρ and r_0 is the density and the radius of the hard body respectively. Besides we put $k_0 = \left(1 - \frac{{r_0}^2}{\hat{R}^2}\right)^{\frac{1}{2}}$ and $k = \left(1 - \frac{r^2}{\hat{R}^2}\right)^{\frac{1}{2}}$ then, equation (1.3) becomes,

$$ds^2 = -k^{-2}\, dr^2 \;\; - r^2 d\theta^2 - r^2 \sin^2\theta\, d\varphi^2 + \left(\frac{3}{2}k_0 - \frac{1}{2}k\right)^2 c^2\, dt^2 \,. \qquad (1.4)$$

The covariant and contravariant metric tensors of equation (1.4) are,

$$g_{\mu\nu} = \begin{pmatrix} -k^{-2} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & c^2\left(\frac{3}{2}k_0 - \frac{1}{2}k\right)^2 \end{pmatrix} \qquad (1.5)$$

and

$$g^{\mu\nu} = \begin{pmatrix} -k^2 & 0 & 0 & 0 \\ 0 & -r^{-2} & 0 & 0 \\ 0 & 0 & -r^{-2}\sin^{-2}\theta & 0 \\ 0 & 0 & 0 & c^{-2}\left(\frac{3}{2}k_0 - \frac{1}{2}k\right)^{-2} \end{pmatrix}. \qquad (1.6)$$

Using coordinate transformations,

$$\begin{cases} r = x \\ \theta = y \\ \varphi = z \\ ct = T. \end{cases} \quad (1.7)$$

2 Formulation of the problem and analysis

Using the interior Schwarzschild space-time metric, the non zeroChristoffel symbols of second kind are,

$$\left\{ {1 \atop 1\ \ 1} \right\} = \frac{x}{x^2-\widehat{R}^2},$$

$$\left\{ {1 \atop 2\ \ 2} \right\} = x\left(1-\frac{x^2}{\widehat{R}^2}\right),$$

$$\left\{ {1 \atop 3\ \ 3} \right\} = x\left(1-\frac{x^2}{\widehat{R}^2}\right)\sin^2 y,$$

$$\left\{ {1 \atop 4\ \ 4} \right\} = -\frac{x^3 + x\widehat{R}^2\left(3\sqrt{1-\frac{x^2}{\widehat{R}^2}}\sqrt{1-\frac{{r_0}^2}{\widehat{R}^2}}-1\right)}{4\widehat{R}^4},$$

$$\left\{ {2 \atop 2\ \ 1} \right\} = -\frac{1}{x},$$

$$\left\{ {2 \atop 3\ \ 3} \right\} = \sin y \cos y,$$

$$\left\{ {3 \atop 3\ \ 1} \right\} = -\frac{1}{x},$$

$$\left\{ {3 \atop 3\ \ 2} \right\} = -\cot y$$

and

$$\left\{ {4 \atop 4\ \ 1} \right\} = \frac{-x}{x^2 + \widehat{R}^2\left(3\sqrt{1-\frac{x^2}{\widehat{R}^2}}\sqrt{1-\frac{{r_0}^2}{\widehat{R}^2}}-1\right)}.$$

Substituting, the non zeroChristoffel symbols in equation (1.2) we get,

$$\frac{\partial\xi_1}{\partial x} + \frac{x}{\widehat{R}^2-x^2}\xi_1 = 0, \quad (2.1)$$

$$\frac{\partial\xi_1}{\partial y} + \frac{\partial\xi_2}{\partial x} + \frac{2}{x}\xi_2 = 0, \quad (2.2)$$

$$\frac{\partial\xi_1}{\partial z} + \frac{\partial\xi_3}{\partial x} + \frac{2}{x}\xi_3 = 0, \quad (2.3)$$

$$\frac{\partial\xi_1}{\partial T} + \frac{\partial\xi_4}{\partial x} + \frac{2x}{x^2 + \widehat{R}^2\left(3\sqrt{1-\frac{x^2}{\widehat{R}^2}}\sqrt{1-\frac{{r_0}^2}{\widehat{R}^2}}-1\right)}\xi_4 = 0, \quad (2.4)$$

$$\frac{\partial\xi_2}{\partial y} - x\left(1-\frac{x^2}{\widehat{R}^2}\right)\xi_1 = 0, \quad (2.5)$$

$$\frac{\partial\xi_2}{\partial z} + \frac{\partial\xi_3}{\partial y} + 2\cot y\ \ \xi_3 = 0, \quad (2.6)$$

$$\frac{\partial\xi_3}{\partial z} - x\left(1-\frac{x^2}{\widehat{R}^2}\right)\sin^2 y\,\xi_1 - \sin y\cos y\,\xi_2 = 0 \quad (2.7)$$

and

$$\frac{\partial\xi_4}{\partial T} + \frac{x^3 + x\widehat{R}^2\left(3\sqrt{1-\frac{x^2}{\widehat{R}^2}}\sqrt{1-\frac{{r_0}^2}{\widehat{R}^2}}-1\right)}{4\widehat{R}^4}\xi_1 = 0. \quad (2.8)$$

With the solution of Killing equations in case of exterior Schwarzschild solution metric in mind [3]; we assume the following situation by guessing we take the first component of the killing vector equal zero,

$$\xi_1 = 0\ . \quad (2.9)$$

Substituting equation (2.9) in equations (2.1) − (2.) we get,

$$\frac{\partial\xi_2}{\partial x} + \frac{2}{x}\xi_2 = 0, \quad (2.10)$$

$$\frac{\partial\xi_3}{\partial x} + \frac{2}{x}\xi_3 = 0, \quad (2.11)$$

$$\frac{\partial\xi_4}{\partial x} + \frac{2x}{x^2 + \widehat{R}^2\left(3\sqrt{1-\frac{x^2}{\widehat{R}^2}}\sqrt{1-\frac{{r_0}^2}{\widehat{R}^2}}-1\right)}\xi_4 = 0, \quad (2.12)$$

$$\frac{\partial\xi_2}{\partial y} = 0, \quad (2.13)$$

$$\frac{\partial\xi_2}{\partial z} + \frac{\partial\xi_3}{\partial y} + 2\cot y\ \ \xi_3 = 0, \quad (2.14)$$

$$\frac{\partial\xi_3}{\partial z} - \sin y\cos y\,\xi_2 = 0 \quad (2.15)$$

and

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$$\frac{\partial \xi_4}{\partial T} = 0\,. \tag{2.16}$$

By guessing, we consider also;

$$\xi_2 = x^{-2}(\Omega_2 \cos z - \Omega_1 \sin z), \tag{2.17}$$

$$\xi_3 = x^{-2}(\Omega_1 \cos z + \Omega_2 \sin z) \sin y \cos y \tag{2.18}$$

and

$$\xi_4 = \Omega_4 \left[\frac{3}{2}\sqrt{1-\frac{{r_0}^2}{\hat{R}^2}} - \frac{1}{2}\sqrt{1-\frac{x^2}{\hat{R}^2}}\right]^{-2}\,. \tag{2.19}$$

Equations $(2.9), (2.17), (2.18)$ and (2.19) represent a solution of the system $(2.1) - (2.8)$ under the condition,

$$y = \frac{n\pi}{4}, \qquad n = 1,2,3, \ldots \ldots \tag{2.20}$$

Using (1.6) in equations $(2.9), (2.17), (2.18)$ and (2.19), the contravariant components of killing vector ξ^α are,

$$\xi^1 = 0, \tag{2.21}$$

$$\xi^2 = -x^{-4}(\Omega_2 \cos z - \Omega_1 \sin z), \tag{2.22}$$

$$\xi^3 = -x^{-4}(\Omega_1 \cos z + \Omega_2 \sin z) \cot y \tag{2.23}$$

and

$$\xi^4 = \Omega_3 \left[\frac{3}{2}\sqrt{1-\frac{{r_0}^2}{\hat{R}^2}} - \frac{1}{2}\sqrt{1-\frac{x^2}{\hat{R}^2}}\right]^{-4}, \tag{2.24}$$

where, Ω_1, Ω_2, Ω_3 and Ω_4 are arbitrary constants.

Existence of killing vector ξ^α leads to existence of conservative quantities due to equation [4,5];

$$m_0 g_{\mu\nu} \frac{dx^\mu}{ds} \xi^\nu_{(n)} = C_{(n)}\,, \tag{2.25}$$

where $C_{(n)}$ are arbitrary constants.

For $n = 1$,

$$m_0 \left(g_{22}\frac{dy}{ds}\xi^2_{(1)} + g_{33}\frac{dz}{ds}\xi^3_{(1)} + g_{44}\frac{dT}{ds}\xi^4_{(1)}\right) = C_{(1)} \tag{2.26}$$

Substituting equations$(1.5), (2.21), (2.22), (2.23)$ and (2.24) in (2.26) we get,

$$m_0 x^{-2}\left(-\frac{dy}{ds}\sin z + \frac{dz}{ds}\sin y \cos y \cos z\right) = C_{(1)}\,, \tag{2.27}$$

Similarly, for $n = 2$ and $n = 4$ we get,

$$m_0 x^{-2}\left(\frac{dy}{ds}\cos z + \frac{dz}{ds}\sin y \cos y \cos z\right) = C_{(2)} \tag{2.28}$$

and

$$m_0 \left[\frac{3}{2}\sqrt{1-\frac{{r_0}^2}{\hat{R}^2}} - \frac{1}{2}\sqrt{1-\frac{x^2}{\hat{R}^2}}\right]^{-2} \frac{dT}{ds} = C_{(4)}\,, \tag{2.29}$$

for $n = 3$ there's no existence of any equation.

3 Conclusion

Under the condition (2.20) which represent a cone, the corresponding conservative quantities in case of interior Schwarzschild metric are, equations (2.27) and (2.28) which represent two component of the angular momentum of free particle. One component of linear momentum is conservative either which represented by equation (2.29).

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